Choquet expectations and g-expectations with multi-dimensional Brownian motion

Mingshang Hu*

School of Mathematics, Shandong University, Jinan, 250100, China

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Abstract. We prove that a g-expectation is a Choquet expectation if and only if g is independent of y and is linear in z, i.e., classical linear expectation, without the assumptions that the deterministic generator g is continuous in t and the dimension of the Brownian motion is one.

Keywords: BSDE, g-expectation, conditional g-expectation, capacity, Choquet expectation, comonotonic additivity.

1 Introduction

Choquet [5] introduced the notion of Choquet expectations via capacities in 1953. Peng [15] introduced the notions of g-expectations and conditional g-expectations via a class of backward stochastic differential equations (BSDEs for short) in 1997. These two types of nonlinear mathematical expectations have their own characteristics. For example, Choquet expectations are comonotonic additivity, g-expectations and conditional g-expectations are consistent. In Chen et al. [2], the authors studied an interesting problem:

If a g-expectation is a Choquet expectation, can we find the form of the generator g?

Under the assumptions that the deterministic generator g is continuous in t and the dimension of the Brownian motion is one, Chen et al. [2] proved that a g-expectation is a Choquet expectation if and only if g is independent

*Email: humingshang@gmail.com

of y and is linear in z. For the case that the dimension of the Brownian motion is greater than one, the main difficulty is to find the form of the generator g. Unfortunately, this problem is not a simple extension of the one dimensional case. Take a 2-dimensional Brownian motion $W_t = (W_t^1, W_t^2)$ for example, W_t^1 and W_t^2 are not comonotonic. This prevents us from using the method in Chen et al. [2] directly. To overcome this defect, we consider comonotonic indicator functions and use a property of BSDE. Furthermore, our method does not need the continuous assumption on g.

This paper is organized as follows: In Section 2, we recall some facts about g-expectations and Choquet expectations. In Section 3, we state and prove our main result.

2 Preliminaries

Fix $T > 0$, let $(W_t)_{0\le t\le T}$ be a d-dimensional standard Brownian motion defined on a completed probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_{0\le t\le T}$ be the natural filtration generated by this Brownian motion. For $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$, $|x| := \sqrt{\sum_{i=1}^d |x_i|^2}$, $x \cdot y := \sum_{i=1}^d x_i y_i$. We denote by $L^2(\mathcal{F}_t)$ the set of all square integrable $\mathcal{F}_t$-measurable random variables and $L^2(0, T; \mathbb{R}^n)$ the space of all $\mathcal{F}_t$-adapted, $\mathbb{R}^n$-valued processes $(v_t)_{t\in[0,T]}$ with $E\int_0^T |v_t|^2 dt < \infty$.

Let us consider a deterministic function $g : [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, which will be in the following the generator of the BSDE. For the function g, we will use the following assumptions:

(H1) For each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, $t \to g(t, y, z)$ is measurable.

(H1′) For each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, $t \to g(t, y, z)$ is continuous.

(H2) There exists a constant $K \ge 0$ such that

$$|g(t, y, z) - g(t, y', z')| \le K(|y - y'| + |z - z'|),\ t \in [0, T], y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d.$$

(H3) $g(t, y, 0) \equiv 0$ for each $(t, y) \in [0, T] \times \mathbb{R}$.

Let g satisfy (H1)-(H3). Then for each $\xi \in L^2(\mathcal{F}_T)$, the BSDE

$$y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s \cdot dW_s,\ 0 \le t \le T, \tag{1}$$

has a unique solution $(y_t,z_t)_{t\in[0,T]} \in L^2(0,T;\mathbb{R}) \times L^2(0,T;\mathbb{R}^d)$ (see Pardoux and Peng [13]), which depends on the generator g and terminal value ξ.

The following standard estimate for BSDEs can be found in [9, 14, 1].

Lemma 1 *Suppose g satisfies (H1)-(H3). For each $\xi^1,\xi^2 \in L^2(\mathcal{F}_T)$, let $(y^i_t,z^i_t)_{t\in[0,T]}$ be the solution of BSDE (1) corresponding to the generator g and terminal value ξ^i with $i=1,2$. Then there exists a constant $C>0$ such that*

$$E[\sup_{t\le s\le T}|y^1_s-y^2_s|^2|\mathcal{F}_t]+E[\int_t^T|z^1_s-z^2_s|^2ds|\mathcal{F}_t] \le CE[|\xi^1-\xi^2|^2|\mathcal{F}_t].$$

Using the solution of BSDE (1), Peng [15] proposed the following notions:

Definition 2 *Suppose g satisfies (H1)-(H3). For each $\xi \in L^2(\mathcal{F}_T)$, let $(y_t,z_t)_{t\in[0,T]}$ be the solution of BSDE (1), define*

$$\mathcal{E}_g[\xi] := y_0; \qquad \mathcal{E}_g[\xi|\mathcal{F}_t] := y_t \quad \textit{for each } t\in[0,T].$$

$\mathcal{E}_g[\xi]$ is called the g-expectation of ξ and $\mathcal{E}_g[\xi|\mathcal{F}_t]$ is called the conditional g-expectation of ξ with respect to $\mathcal{F}_t$.

We now recall the notions of capacity and Choquet expectation. A capacity is a set function $V:\mathcal{F}_T \mapsto [0,1]$ satisfying: (i) $V(\emptyset)=0$, $V(\Omega)=1$; (ii) $V(A)\le V(B)$ for each $A\subset B$. The corresponding Choquet expectation (see [5]) is defined as follows:

$$\mathcal{C}[\xi] := \int_{-\infty}^0 [V(\xi\ge t)-1]dt + \int_0^\infty V(\xi\ge t)dt \quad \text{for each } \xi\in L^2(\mathcal{F}_T).$$

Two random variables ξ and η are called comonotonic if

$$[\xi(\omega)-\xi(\omega')][\eta(\omega)-\eta(\omega')] \ge 0 \quad \text{for each } \omega,\omega'\in\Omega.$$

Now, we list some properties of Choquet expectations (see [5, 16, 7, 8]).

(1) Monotonicity: If $\xi\ge\eta$, then $\mathcal{C}[\xi]\ge\mathcal{C}[\eta]$.

(2) Positive homogeneity: If $\lambda\ge 0$, then $\mathcal{C}[\lambda\xi]=\lambda\mathcal{C}[\xi]$.

(3) Translation invariance: If $c\in\mathbb{R}$, then $\mathcal{C}[\xi+c]=\mathcal{C}[\xi]+c$.

(4) Comonotonic additivity: If ξ and η are comonotonic, then $\mathcal{C}[\xi+\eta] = \mathcal{C}[\xi] + \mathcal{C}[\eta]$.

Let g satisfy (H1)-(H3), define

$$P_g(A) := \mathcal{E}_g[I_A] \quad \text{for each } A \in \mathcal{F}_T.$$

$P_g(A)$ is called the g-probability of A. Obviously, $P_g(\cdot)$ is a capacity. The corresponding Choquet expectation is denoted by $\mathcal{C}_g$. It is easy to check that $\mathcal{C}_g[I_A] = \mathcal{E}_g[I_A]$ for each $A \in \mathcal{F}_T$. Furthermore, $\mathcal{C}_g[\xi] < \infty$ for each $\xi \in L^2(\mathcal{F}_T)$ (see [10]).

The following result can be found in [2].

Lemma 3 *Suppose that $d = 1$ and g satisfies (H1′), (H2) and (H3). Then $\mathcal{E}_g[\xi] = \mathcal{C}_g[\xi]$ for each $\xi \in L^2(\mathcal{F}_T)$ if and only if g is independent of y and is linear in z, i.e., $g(t,z) = g(t,1)z$.*

3 Main result

Let $\{e_1, e_2, \ldots, e_d\}$ denote the standard basis of $\mathbb{R}^d$. Now we give the main result.

Theorem 4 *Suppose g satisfies (H1)-(H3). Then $\mathcal{E}_g[\xi] = \mathcal{C}_g[\xi]$ for each $\xi \in L^2(\mathcal{F}_T)$ if and only if g is independent of y and is linear in z, i.e., $g(t,z) = \sum_{i=1}^d g(t,e_i)z_i$ for almost every $t \in [0,T]$, where z_i is the i-th component of z.*

For proving this theorem, we need the following lemmas. The first lemma is a direct consequence of Jiang [12] (see also [1, 2, 11]).

Lemma 5 *Suppose g satisfies (H1)-(H3). If $\mathcal{E}_g[\xi] = \mathcal{C}_g[\xi]$ for each $\xi \in L^2(\mathcal{F}_T)$, then g is independent of y and is positively homogeneous in z.*

Proof. Since $\mathcal{E}_g = \mathcal{C}_g$, we have

$$\mathcal{E}_g[\xi + c] = \mathcal{E}_g[\xi] + c \quad \text{for each } c \in \mathbb{R};\ \mathcal{E}_g[\lambda\xi] = \lambda\mathcal{E}_g[\xi] \quad \text{for each } \lambda \geq 0.$$

From this, we obtain the result (see Theorems 3.1 and 3.4 in Jiang [12]). The proof is complete. $\square$

The next lemma is a property of BSDE (see [14]).

Lemma 6 *Suppose g satisfies (H1)-(H3). Let ξ be a k_1-dimensional $\mathcal{F}_{t_0}$-measurable random vector and η be a k_2-dimensional $\mathcal{F}_T$-measurable random vector, where $t_0 \in [0,T)$ and $k_1, k_2 \in \mathbb{N}$. Then for each $f \in C_b(\mathbb{R}^{k_1} \times \mathbb{R}^{k_2})$, we have*

$$\mathcal{E}_g[f(\xi,\eta)|\mathcal{F}_t] = \mathcal{E}_g[f(x,\eta)|\mathcal{F}_t]|_{x=\xi},\ t \in [t_0,T].$$

Proof. We outline the proof for the convenience of the reader. The proof is divided into two steps.

Step 1: Let ξ be simple random vector, i.e., $\xi = \sum_{i=1}^N x_i I_{A_i}$, where $\{x_i\}_{i=1}^N \subset \mathbb{R}^{k_1}$ and $\{A_i\}_{i=1}^N$ is an $\mathcal{F}_{t_0}$-partition of Ω. Let $(y_t^i, z_t^i)_{t\in[0,T]}$ denote the solution of BSDE (1) corresponding to the generator g and terminal value $f(x_i,\eta)$ with $i = 1,\ldots,N$. Then it is easy to verify that $(\sum_{i=1}^N y_t^i I_{A_i}, \sum_{i=1}^N z_t^i I_{A_i})_{t\in[t_0,T]}$ is the solution of BSDE (1) on $[t_0,T]$ corresponding to the generator g and terminal value $\sum_{i=1}^N f(x_i,\eta) I_{A_i}$. Noting that $f(\sum_{i=1}^N x_i I_{A_i}, \eta) = \sum_{i=1}^N f(x_i,\eta) I_{A_i}$, then for $t \in [t_0,T]$, we have

$$\mathcal{E}_g[f(\xi,\eta)|\mathcal{F}_t] = \sum_{i=1}^N \mathcal{E}_g[f(x_i,\eta)|\mathcal{F}_t] I_{A_i} = \mathcal{E}_g[f(x,\eta)|\mathcal{F}_t]|_{x=\xi}.$$

Step 2: For general ξ, we can choose some simple random vectors $\xi_n \to \xi$. Since $f \in C_b(\mathbb{R}^{k_1} \times \mathbb{R}^{k_2})$, by Lemma 1, we get for $t \in [t_0,T]$,

$$P\text{--}a.s.,\ \mathcal{E}_g[f(\xi_n,\eta)|\mathcal{F}_t] \to \mathcal{E}_g[f(\xi,\eta)|\mathcal{F}_t], \mathcal{E}_g[f(x,\eta)|\mathcal{F}_t]|_{x=\xi_n} \to \mathcal{E}_g[f(x,\eta)|\mathcal{F}_t]|_{x=\xi}.$$

Thus $\mathcal{E}_g[f(\xi,\eta)|\mathcal{F}_t] = \mathcal{E}_g[f(x,\eta)|\mathcal{F}_t]|_{x=\xi}$. The proof is complete. $\square$

Remark 7 *Let $f_n \in C_b(\mathbb{R}^{k_1} \times \mathbb{R}^{k_2})$ be uniformly bounded such that $f_n \to f$. Then by Lemma 1, we can easily prove that Lemma 6 still holds for f.*

The following lemma plays an important role in proving the main theorem with $d = 1$.

Lemma 8 *Suppose that $d = 1$ and g satisfies (H1)-(H3). If $\mathcal{E}_g[\xi] = \mathcal{C}_g[\xi]$ for each $\xi \in L^2(\mathcal{F}_T)$, then for each $t \in [0,T]$, $n \in \mathbb{N}$, we have*

$$\mathcal{E}_g[I_{[W_T \geq -n]} + I_{[0 \geq W_T \geq -n]}|\mathcal{F}_t] = \mathcal{E}_g[I_{[W_T \geq -n]}|\mathcal{F}_t] + \mathcal{E}_g[I_{[0 \geq W_T \geq -n]}|\mathcal{F}_t].$$

Proof. Let $W_{t,T}$ denote $W_T - W_t$. For each $a < b$, it is easy to verify that $I_{[W_{t,T} \geq a]}$ and $I_{[b \geq W_{t,T} \geq a]}$ are comonotonic. Then, by $\mathcal{E}_g = \mathcal{C}_g$ and the comonotonic additivity of the Choquet expectation, we have

$$\mathcal{E}_g[I_{[W_{t,T} \geq a]} + I_{[b \geq W_{t,T} \geq a]}] = \mathcal{E}_g[I_{[W_{t,T} \geq a]}] + \mathcal{E}_g[I_{[b \geq W_{t,T} \geq a]}]. \tag{2}$$

On the other hand, for each $l_1, l_2 \in \mathbb{R}$, it is easy to show that $f(x,y) := l_1 I_{[x+y \geq -n]} + l_2 I_{[0 \geq x+y \geq -n]}$ satisfies the condition in Remark 7. Hence, we have

$$\mathcal{E}_g[l_1 I_{[W_T \geq -n]} + l_2 I_{[0 \geq W_T \geq -n]} | \mathcal{F}_t] = \mathcal{E}_g[l_1 I_{[W_{t,T} \geq -n-\bar{a}]} + l_2 I_{[-\bar{a} \geq W_{t,T} \geq -n-\bar{a}]}]|_{\bar{a}=W_t}. \tag{3}$$

Combining (3) with (2) yields the result, and the proof is complete. □

The following lemma is our main theorem with $d = 1$, which is an extension of Lemma 3.

Lemma 9 *Suppose that $d = 1$ and g satisfies (H1)-(H3). Then $\mathcal{E}_g[\xi] = \mathcal{C}_g[\xi]$ for each $\xi \in L^2(\mathcal{F}_T)$ if and only if g is independent of y and is linear in z, i.e., $g(t,z) = g(t,1)z$ for almost every $t \in [0,T]$.*

Proof. If $g(t,z) = g(t,1)z$, by the Girsanov Theorem, the g-expectation is a linear mathematical expectation. Therefore, $\mathcal{E}_g[\xi] = \mathcal{C}_g[\xi]$ for each $\xi \in L^2(\mathcal{F}_T)$ and the proof of sufficient condition is complete. Now we prove the necessary condition. By Lemma 5, g is independent of y. For each $n \in \mathbb{N}$, consider the following BSDEs:

$$\begin{aligned}
y_t^n &= I_{[W_T \geq -n]} + I_{[0 \geq W_T \geq -n]} + \int_t^T g(s, z_s^n) ds - \int_t^T z_s^n dW_s, \\
\tilde{y}_t^n &= I_{[W_T \geq -n]} + \int_t^T g(s, \tilde{z}_s^n) ds - \int_t^T \tilde{z}_s^n dW_s, \\
\hat{y}_t^n &= I_{[0 \geq W_T \geq -n]} + \int_t^T g(s, \hat{z}_s^n) ds - \int_t^T \hat{z}_s^n dW_s.
\end{aligned}$$

By Lemma 8, we have $y_t^n = \tilde{y}_t^n + \hat{y}_t^n$ for each $t \in [0,T]$. Form this, we have

$$dP \times dt - a.s., \quad g(t, \tilde{z}_t^n + \hat{z}_t^n) = g(t, \tilde{z}_t^n) + g(t, \hat{z}_t^n). \tag{4}$$

On the other hand, it follows from Lemma 5 that g is positively homogeneous. Thus we have for almost every $t \in [0,T]$,

$$g(t,z) = g(t,1)z^+ + g(t,-1)z^-, \tag{5}$$

where $z^+ = \max\{z, 0\}$, $z^- = (-z)^+$. Set $h(t) := g(t, 1) + g(t, -1)$, by (4) and (5), we have

$$dP \times dt - a.s., \quad h(t)(\tilde{z}_t^n + \hat{z}_t^n)^- = h(t)(\tilde{z}_t^n)^- + h(t)(\hat{z}_t^n)^-. \qquad (6)$$

Also, $dP \times dt - a.s.$, $\tilde{z}_t^n = \frac{1}{\sqrt{2\pi(T-t)}} \exp(-\frac{(n+W_t+\int_t^T g(s,1)ds)^2}{2(T-t)}) > 0$ (see Lemma 8 in [2, Chen et al. (2005a)]). This with (6) implies

$$dP \times dt - a.s., \quad h(t) I_{[\hat{z}_t^n < 0]} = 0. \qquad (7)$$

Let $(\bar{y}_t, \bar{z}_t)_{t\in[0,T]}$ denote the solution of BSDE (1) corresponding to the generator g and terminal value $I_{[W_T \leq 0]}$. It follows from Lemma 1 that $\hat{z}_t^n \to \bar{z}_t$ as $n \to \infty$ in $L^2(0, T; \mathbb{R})$. Thus we can choose $n_i \to \infty$ such that $dP \times dt - a.s.$, $\hat{z}_t^{n_i} \to \bar{z}_t$. Noting that $\bar{z}_t = \frac{-1}{\sqrt{2\pi(T-t)}} \exp(-\frac{(W_t - \int_t^T g(s,-1)ds)^2}{2(T-t)}) < 0$, then by (7), we can deduce that for almost every $t \in [0, T]$, $h(t) = 0$. Thus $g(t, z) = g(t, 1)z$ for almost every $t \in [0, T]$. The proof is complete. $\square$

Corollary 10 *Suppose g satisfies (H1)-(H3). If $\mathcal{E}_g[\xi] = \mathcal{C}_g[\xi]$ for each $\xi \in L^2(\mathcal{F}_T)$, then g is independent of y and is homogeneous in z, i.e., for almost every $t \in [0, T]$, $g(t, \lambda z) = \lambda g(t, z)$ for each $\lambda \in \mathbb{R}$.*

Proof. For each fixed $a \in \mathbb{R}^d$ with $|a| = 1$, set $\tilde{W}_t := a \cdot W_t$ and $\tilde{\mathcal{F}}_t := \sigma\{\tilde{W}_s : s \leq t\}$ for each $t \in [0, T]$. Obviously, $(\tilde{W}_t)_{t\in[0,T]}$ is a 1-dimensional Brownian motion. Define $\tilde{g} : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\tilde{g}(t, y, z) := g(t, y, az)$. It is easy to verify that $\tilde{g}$ satisfies (H1)-(H3). For each $\xi \in L^2(\tilde{\mathcal{F}}_T)$, let $(y_t, z_t)_{t\in[0,T]}$ denote the solution of the following BSDE:

$$y_t = \xi + \int_t^T \tilde{g}(s, y_s, z_s)ds - \int_t^T z_s d\tilde{W}_s, \ 0 \leq t \leq T.$$

Then it is easy to check that $(y_t, az_t)_{t\in[0,T]}$ is the solution of BSDE (1) corresponding to the generator g and terminal value ξ. From this, we deduce that $\mathcal{E}_g[\xi] = \mathcal{E}_{\tilde{g}}[\xi]$ for each $\xi \in L^2(\tilde{\mathcal{F}}_T)$. Noting that $\mathcal{E}_g = \mathcal{C}_g$, we then get $\mathcal{E}_{\tilde{g}}[\xi] = \mathcal{C}_{\tilde{g}}[\xi]$ for each $\xi \in L^2(\tilde{\mathcal{F}}_T)$. By Lemma 9, we obtain $\tilde{g}(t, y, z) = \tilde{g}(t, 0, 1)z$ for almost every $t \in [0, T]$. Hence, by the Lipschitz assumption (H2), we have for almost every $t \in [0, T]$, $g(t, y, \lambda a) = \lambda g(t, 0, a)$ for each $\lambda \in \mathbb{R}$ and $a \in \mathbb{R}^d$ with $|a| = 1$, which implies that g is independent of y and is homogeneous in z. We complete the proof. $\square$

To prove the main theorem, we also pay more attention to the following two lemmas.

Lemma 11 *Suppose that $d = 2$ and g satisfies (H1)-(H3). If $\mathcal{E}_g[\xi] = \mathcal{C}_g[\xi]$ for each $\xi \in L^2(\mathcal{F}_T)$, then for each $\lambda \in [0,1]$, $t \in [0,T]$, $n \in \mathbb{N}$, we have*

$$\mathcal{E}_g[I_{[W_T^1 \geq n]} + \lambda I_{[W_T^2 \geq 0]} | \mathcal{F}_t] = \lambda \mathcal{E}_g[I_{[W_T^1 \geq n]} + I_{[W_T^2 \geq 0]} | \mathcal{F}_t] + (1-\lambda)\mathcal{E}_g[I_{[W_T^1 \geq n]} | \mathcal{F}_t],$$

where W_t^i is the i-th component of W_t with $i = 1, 2$.

Proof. Let $W_{t,T}^i$ denote $W_T^i - W_t^i$ with $i = 1, 2$. For each fixed $\lambda \in [0,1]$, $a, b \in \mathbb{R}$, it is easy to check that $(1-\lambda)I_{[W_{t,T}^1 \geq a]}$ and $\lambda(I_{[W_{t,T}^1 \geq a]} + I_{[W_{t,T}^2 \geq b]})$ are comonotonic. The rest of the proof runs as in Lemma 8, and the proof is complete. $\square$

Lemma 12 *Suppose that $d = 2$ and g satisfies (H1)-(H3). If $\mathcal{E}_g[\xi] = \mathcal{C}_g[\xi]$ for each $\xi \in L^2(\mathcal{F}_T)$, then g is independent of y and is linear in z, i.e., $g(t, z_1, z_2) = g(t, 1, 0)z_1 + g(t, 0, 1)z_2$ for almost every $t \in [0,T]$.*

Proof. It follows from Lemma 5 that g is independent of y. For each fixed $\lambda \in (0,1)$, $n \in \mathbb{N}$, consider the following BSDEs:

$$\begin{aligned}
y_t^{\lambda,n} &= I_{[W_T^1 \geq n]} + \lambda I_{[W_T^2 \geq 0]} + \int_t^T g(s, z_{1,s}^{\lambda,n}, z_{2,s}^{\lambda,n})ds - \int_t^T z_{1,s}^{\lambda,n} dW_s^1 - \int_t^T z_{2,s}^{\lambda,n} dW_s^2, \\
\tilde{y}_t^n &= I_{[W_T^1 \geq n]} + I_{[W_T^2 \geq 0]} + \int_t^T g(s, \tilde{z}_{1,s}^n, \tilde{z}_{2,s}^n)ds - \int_t^T \tilde{z}_{1,s}^n dW_s^1 - \int_t^T \tilde{z}_{2,s}^n dW_s^2, \\
\hat{y}_t^n &= I_{[W_T^1 \geq n]} + \int_t^T g(s, \hat{z}_{1,s}^n, \hat{z}_{2,s}^n)ds - \int_t^T \hat{z}_{1,s}^n dW_s^1 - \int_t^T \hat{z}_{2,s}^n dW_s^2.
\end{aligned}$$

By Lemma 11, we have $y_t^{\lambda,n} = \lambda \tilde{y}_t^n + (1-\lambda)\hat{y}_t^n$ for each $t \in [0,T]$. From this, we deduce that $dP \times dt - a.s.$,

$$g(t, \lambda \tilde{z}_{1,t}^n + (1-\lambda)\hat{z}_{1,t}^n, \lambda \tilde{z}_{2,t}^n + (1-\lambda)\hat{z}_{2,t}^n) = \lambda g(t, \tilde{z}_{1,t}^n, \tilde{z}_{2,t}^n) + (1-\lambda) g(t, \hat{z}_{1,t}^n, \hat{z}_{2,t}^n).$$

Since $\lambda \in (0,1)$ is arbitrary, by Lemma 5, we obtain that $dP \times dt - a.s.$,

$$g(t, \tilde{z}_{1,t}^n + l\hat{z}_{1,t}^n, \tilde{z}_{2,t}^n + l\hat{z}_{2,t}^n) = g(t, \tilde{z}_{1,t}^n, \tilde{z}_{2,t}^n) + g(t, l\hat{z}_{1,t}^n, l\hat{z}_{2,t}^n) \text{ for each } l \geq 0. \quad (8)$$

Noting that $g(t, z_1, 0) = g(t, 1, 0)z_1$ for almost every $t \in [0,T]$, then we have

$$dP \times dt - a.s., \ (\hat{z}_{1,t}^n, \hat{z}_{2,t}^n) = \left(\frac{1}{\sqrt{2\pi(T-t)}} \exp\left(-\frac{(n - W_t^1 - \int_t^T g(s,1,0)ds)^2}{2(T-t)}\right), 0\right). \quad (9)$$

Combining (8) with (9), we get

$$dP\times dt-a.s.,\ g(t,\tilde{z}^n_{1,t}+p,\tilde{z}^n_{2,t})=g(t,\tilde{z}^n_{1,t},\tilde{z}^n_{2,t})+g(t,p,0) \text{ for each } p\geq 0. \quad (10)$$

Let $(\bar{y}_t,\bar{z}_{1,t},\bar{z}_{2,t})_{t\in[0,T]}$ be the solution of BSDE (1) corresponding to the generator g and terminal value $I_{[W^2_T\geq 0]}$. By Lemma 1, we have $(\tilde{z}^n_{1,t},\tilde{z}^n_{2,t})\to(\bar{z}_{1,t},\bar{z}_{2,t})$ in $L^2(0,T;\mathbb{R}^2)$. Since g satisfies Lipschitz assumption (H2), we get for each $p\geq 0$,

$$g(t,\tilde{z}^n_{1,t}+p,\tilde{z}^n_{2,t})\to g(t,\bar{z}_{1,t}+p,\bar{z}_{2,t}) \text{ in } L^2(0,T;\mathbb{R}).$$

This with (10) implies that

$$dP\times dt-a.s.,\ g(t,\bar{z}_{1,t}+p,\bar{z}_{2,t})=g(t,\bar{z}_{1,t},\bar{z}_{2,t})+g(t,p,0) \text{ for each } p\geq 0. \quad (11)$$

Also, we have

$$dP\times dt-a.s.,\ (\bar{z}_{1,t},\bar{z}_{2,t})=(0,\frac{1}{\sqrt{2\pi(T-t)}}\exp(-\frac{(W^2_t+\int_t^T g(s,0,1)ds)^2}{2(T-t)})). \quad (12)$$

It follows from (11), (12) and Lemma 5 that for almost every $t\in[0,T]$,

$$g(t,p,1)=g(t,0,1)+g(t,p,0) \text{ for each } p\geq 0. \quad (13)$$

From (13) and Corollary 10, we can easily deduce that for almost every $t\in[0,T]$,

$$g(t,z_1,z_2)=g(t,1,0)z_1+g(t,0,1)z_2 \text{ for each } z_1\cdot z_2\geq 0. \quad (14)$$

On the other hand, set $\tilde{W}_t:=(W^1_t,-W^2_t)$ and $\tilde{g}(t,z_1,z_2)=g(t,z_1,-z_2)$. Analysis similar to that in the proof of Corollary 10 shows that $\mathcal{E}_{\tilde{g}}[\xi]=\mathcal{C}_{\tilde{g}}[\xi]$ for each $\xi\in L^2(\mathcal{F}_T)$. Then we have (14) for $\tilde{g}$, which gives that for almost every $t\in[0,T]$,

$$g(t,z_1,z_2)=g(t,1,0)z_1+g(t,0,1)z_2 \text{ for each } z_1\cdot z_2\leq 0.$$

The proof is now complete. $\square$

We now prove the main theorem.

Proof of Theorem 4. The sufficient condition can be proved by the same method as in Lemma 9. We only prove the necessary condition. For

$d = 2$, by Lemma 12, the result holds. We only prove the case $d > 2$. For each fixed $a \in \mathbb{R}^{d-1}$ with $|a| = 1$, set $\tilde{W}_t := (a \cdot (W_t^1, \ldots, W_t^{d-1}), W_t^d)$ and $\tilde{\mathcal{F}}_t := \sigma\{\tilde{W}_s : s \leq t\}$ for each $t \in [0, T]$. By Lemma 5, g is independent of y, we define $\tilde{g} : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\tilde{g}(t, z_1, z_2) := g(t, az_1, z_2)$. As in the proof of Corollary 10, we can get $\mathcal{E}_{\tilde{g}}[\xi] = \mathcal{C}_{\tilde{g}}[\xi]$ for each $\xi \in L^2(\tilde{\mathcal{F}}_T)$. By Lemma 12, we have for almost every $t \in [0, T]$,

$$\tilde{g}(t, z_1, z_2) = \tilde{g}(t, 1, 0)z_1 + \tilde{g}(t, 0, 1)z_2.$$

Since a is arbitrary, by Corollary 10, we obtain for almost every $t \in [0, T]$,

$$g(t, z_1, \ldots, z_{d-1}, z_d) = g(t, z_1, \ldots, z_{d-1}, 0) + g(t, e_d)z_d.$$

Define $\bar{g} : [0, T] \times \mathbb{R}^{d-1} \to \mathbb{R}$ by $\bar{g}(t, z) := g(t, z, 0)$. We now apply the above argument again, with g replaced by $\bar{g}$, to obtain that for almost every $t \in [0, T]$,

$$\bar{g}(t, z_1, \ldots, z_{d-2}, z_{d-1}) = \bar{g}(t, z_1, \ldots, z_{d-2}, 0) + \bar{g}(t, 0, \ldots, 0, 1)z_{d-1},$$

that is

$$g(t, z_1, \ldots, z_{d-2}, z_{d-1}, 0) = g(t, z_1, \ldots, z_{d-2}, 0, 0) + g(t, e_{d-1})z_{d-1}.$$

Continuing this process, we can prove that $g(t, z) = \sum_{i=1}^{d} g(t, e_i)z_i$ for almost every $t \in [0, T]$. The proof is complete. $\square$

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